

21-241 Exam 1 Solutions

1. Determine whether the following statements are true or false. If a statement is true, prove it. If it is false, explain why or provide a counterexample.
- (a) If A is invertible and $AB = A$, then $B = I$.

Solution. True. We left multiply both sides by A^{-1} and apply associativity:

$$\begin{aligned}AB &= A \\A^{-1}(AB) &= A^{-1}A \\(A^{-1}A)B &= I \\IB &= I \\B &= I\end{aligned}$$

- (b) Let A be an $m \times n$ matrix. If $C(A) = \mathbb{R}^m$ and $N(A) = \{\mathbf{0}\}$, then $m = n$.

Solution (one of many). True. Since $N(A) = \{\mathbf{0}\}$, the columns of A are linearly independent. Moreover, we are given that the columns of A span $\mathbb{R}^m$. Taking these two observations together, we have that the columns of A form a *basis* for $\mathbb{R}^m$. Every basis for $\mathbb{R}^m$ must have m vectors, so the number of columns of A , n , is equal to m .

- (c) The subspace $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right\}$ of $\mathbb{R}^3$ has dimension 2.

Solution. False. The second vector is a scalar multiple of the first. Thus, the subspace spanned by the two vectors is in fact also spanned by only the first vector. So the subspace has dimension 1.

2. Suppose a matrix A has RREF $\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$. Answer the following questions and explain your reasoning.

- (a) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b} \in \mathbb{R}^3$?

Solution. Yes. Since the RREF of A has no zero rows (that is, each row has a pivot), the system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

(b) Are the columns of A linearly independent?

Solution. No. Since there is not a pivot in every column, the columns are not linearly independent.

(c) What are the dimensions of the four fundamental subspaces?

Solution. The dimension of each of $C(A)$ and $C(A^T)$ is equal to the number of pivots $r = 3$. The dimension of $N(A)$ is equal to $n - r = 5 - 3 = 2$. The dimension of $N(A^T)$ is equal to $m - r = 3 - 3 = 0$, that is, $N(A^T)$ is trivial.

(d) Find a basis for $N(A)$.

Solution. A basis for $N(A)$ is given by the special solutions associated to the RREF. To find these, we write the pivot variables x_1, x_3, x_4 in terms of the free variables x_2, x_5 :

$$x_1 = -2x_2 - 2x_5$$

$$x_3 = x_5$$

$$x_4 = -3x_5.$$

So a general solution is of the form

$$\begin{pmatrix} -2x_2 - 2x_5 \\ x_2 \\ x_5 \\ -3x_5 \\ x_5 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \\ -3 \\ 1 \end{pmatrix}$$

thus a basis for $N(A)$ is given by

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

- (a) Reduce A to RREF. Be sure to explain your steps.

Solution.

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} &\xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{R_2 \rightarrow R_2 / (-2)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So the RREF of A is given by

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- (b) Find a basis for $C(A)$. Explain your reasoning.

Solution. To find a basis for $C(A)$, extract the columns of A corresponding to the pivot columns of the RREF of A (columns 1 and 2):

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

- (c) What does $C(A)$ look like, geometrically? Explain your reasoning.

Solution. Since $C(A)$ is a 2-dimensional subspace of $\mathbb{R}^3$ (by part (b)), it looks like a plane in $\mathbb{R}^3$ through the origin.

- (d) Is $\mathbf{b} \in C(A)$? If so, express it as a linear combination of the basis vectors of $C(A)$. If not, explain why.

Solution. By definition, $\mathbf{b} \in C(A)$ if and only if $A\mathbf{x} = \mathbf{b}$ has at least one solution. To determine this, we must reduce the augmented matrix $[A \mid \mathbf{b}]$. Since we already reduced A in part (a), we can use the same row operations to reduce $\mathbf{b}$:

$$\begin{aligned} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} &\xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 3R_1} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &\xrightarrow{R_2 \rightarrow R_2 / (-2)} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ &\xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence the RREF of $[A \mid \mathbf{b}]$ is

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 3 \\ 0 & 1 & 2 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

We see that the system has a solution because the final column does not contain a pivot. To express $\mathbf{b}$ as a linear combination of the basis vectors, we must first find a solution to $A\mathbf{x} = \mathbf{b}$. One such solution is obtained by setting the free variables equal to 0 and finding that $x_1 = 3$ and $x_2 = -1$. So expressing $\mathbf{b}$ as a linear combination of the basis vectors found in part (b), we have:

$$\mathbf{b} = 3 \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}.$$

4. Consider the map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 2y \\ 0 \end{bmatrix}$$

(a) Show that T is a linear transformation using the definition of linear transformations.

Solution. Let $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$. Then

$$\begin{aligned} T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) &= T \left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ 2(y_1 + y_2) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - z_1 \\ 2y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ 2y_2 \\ 0 \end{bmatrix} \\ &= T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) + T \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) \end{aligned}$$

and for $c \in \mathbb{R}$,

$$T \left(c \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) = T \left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix} \right) = \begin{bmatrix} cx_1 - cz_1 \\ 2cy_1 \\ 0 \end{bmatrix} = c \cdot T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right).$$

(b) Find the matrix associated to T . Explain your reasoning.

Solution. The matrix A associated to T is the (unique) matrix A satisfying $A\mathbf{v} = T(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^3$. Since

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 2y \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the matrix is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Find a basis for the image (i.e. range) of T .

Solution. The image of T is precisely the column space of A . Since A is already in upper triangular form, we can take a basis for $C(A)$ to be the set of pivot columns, that is,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

(d) Find the matrix associated to T^{-1} , if it exists. If it does not exist, explain why.

Solution. T is invertible if and only if A is invertible, in which case A^{-1} is the matrix associated to T^{-1} . Since A has fewer than 3 pivots, it is not invertible, so T^{-1} does not exist.

5. Suppose $\{\mathbf{v}, \mathbf{w}\} \subset \mathbb{R}^n$ is linearly independent. Use the definition of linear independence to show that $\{\mathbf{v}, a\mathbf{w} + b\mathbf{v}\}$ is linearly independent if $a \neq 0$.

Solution. We need to show that if $c_1, c_2 \in \mathbb{R}$ and $c_1\mathbf{v} + c_2(a\mathbf{w} + b\mathbf{v}) = \mathbf{0}$, then $c_1 = c_2 = 0$.

$$\begin{aligned} c_1\mathbf{v} + c_2(a\mathbf{w} + b\mathbf{v}) &= \mathbf{0} \\ \implies (c_1 + c_2b)\mathbf{v} + c_2a\mathbf{w} &= \mathbf{0} \end{aligned}$$

But since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, we must have that

$$c_1 + c_2b = 0 \quad \text{and} \quad c_2a = 0$$

but since $a \neq 0$, we have that $c_2 = 0$, and thus, by substituting into the first equation, we have that $c_1 = 0$, which is what we sought to prove.